Available online at www.sciencedirect.com

ScienceDirect

Journal of Symbolic Computation 42 (2007) 1052–1065

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Poincaré–Birkhoff–Witt theorem for Leibniz n-algebras

José Manuel Casas[a], Manuel A. Insua[b], Manuel Ladra[b,*]

[a] *Dpto. Matemática Aplicada I, Universidad de Vigo, E-36005 Pontevedra, Spain*
[b] *Departamento de Álgebra, Universidad de Santiago, E-15782, Spain*

Received 11 December 2006; accepted 2 May 2007
Available online 1 September 2007

Abstract

We construct the universal enveloping algebra of a Leibniz n-algebra and we prove that the category of modules over this algebra is equivalent to the category of representations.

We also give a proof of the Poincaré–Birkhoff–Witt theorem for universal enveloping algebras of finite-dimensional Leibniz n-algebras using Gröbner bases in a free associative algebra.

Keywords: Leibniz n-algebra; Universal enveloping algebra; Poincaré–Birkhoff–Witt theorem; Gröbner bases

1. Introduction

In the framework of Hamiltonian mechanics the equations of motion are given by means of the Hamiltonian operator H, $\frac{\mathrm{d}f}{\mathrm{d}t} = \{H, f\}$, where $\{-, -\}$ denotes the classical Poisson bracket which is defined by $\{f_1, f_2\} = \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}$ for a function f defined on $\mathbb{R}^2$.

In 1973, Nambu (1973) proposed the natural generalization of the last equations from a binary bracket to a ternary bracket, and in general to an n-ary bracket for functions defined on $\mathbb{R}^n$, given by the jacobian of the function $f = (f_1, \ldots, f_n)$. The Nambu–Hamilton generalized equations of the motion include $n - 1$ Hamiltonian operators and the n-ary bracket satisfies a generalized Jacobi identity. So a new kind of n-ary algebras was born, the so-called Nambu–Lie algebras. This structure was also introduced by Filippov (1985) in the framework of the geometry.

* Corresponding author. Tel.: +34 981563100; fax: +34 981597054.
E-mail addresses: jmcasas@uvigo.es (J.M. Casas), avelino.insua@gmail.com (M.A. Insua), ladra@usc.es (M. Ladra).
URL: http://web.usc.es/~mladra/ (M. Ladra).

0747-7171/$ - see front matter
doi:10.1016/j.jsc.2007.05.003

Nevertheless, in the 90s, Loday (1993) introduced a non-skew-symmetric version of Lie algebras, the so-called Leibniz algebras. This algebraic structure had a prominent development from the 90s and it was the subject of a lot of papers in various fields: algebra, geometry, physics, etc. (Ibáñez et al., 1999; Kinyon and Weinstein, 2001; Hagiwara and Mizutani, 2002). Motivated by these ideas, in 2002, Casas et al. (2002) introduced the n-ary version of Leibniz algebras, called Leibniz n-algebras, whose skew-symmetric counterparts are the Nambu–Lie or Filippov algebras. The main examples of Leibniz n-algebras which are not Lie n-algebras are the Lie triple systems (Jacobson, 1949; Lister, 1952).

The construction of the universal enveloping algebra $\mathrm{UL}(\mathfrak{g})$ of a Leibniz algebra $\mathfrak{g}$ was given by Loday and Pirashvili (1993), where they show that the category of representations of a Leibniz algebra $\mathfrak{g}$ is equivalent to the category of right modules over $\mathrm{UL}(\mathfrak{g})$ and they also give a Poincaré–Birkhoff–Witt theorem for this kind of algebras. In Insua and Ladra (submitted for publication) a different proof of this theorem is given using Gröbner bases. The proof of the PBW theorem for Lie algebras using Gröbner bases was given by Bergman (1978). The representation theory of Lie triple systems was given in Jacobson (1949), Harris (1961) and Hodge and Parshall (2002) where they introduced the construction of a universal enveloping algebra for a Lie triple system and a PBW type theorem.

The aim of the present paper is to describe the representations of Leibniz n-algebras by means of a universal enveloping algebra and to establish a PBW type theorem that allows us to make calculations in a similar way as in a ring of polynomials $K[x_1, \ldots, x_n]$.

The paper is structured as follows. In Section 2 we recall basic results on Leibniz n-algebras. In Section 3 we construct the universal enveloping algebra $\mathrm{U_nL}(\mathcal{L})$ of a Leibniz n-algebra $\mathcal{L}$ and we prove that the category of representations of a Leibniz n-algebra $\mathcal{L}$ is equivalent to the category of right modules over $\mathrm{U_nL}(\mathcal{L})$. In Section 4 we give a proof of a Poincaré–Birkhoff–Witt theorem for universal enveloping algebras of finite-dimensional Leibniz n-algebras using Gröbner bases in a free associative algebra. In the last section we write a program in NCAlgebra (a package running under Mathematica) which calculates Gröbner bases of the ideal that determines $\mathrm{U_nL}(\mathcal{L})$, for small values of n and low dimensions of $\mathcal{L}$.

2. Leibniz n-algebras

Definition 1. A *Leibniz n-algebra* (Casas et al., 2002) over a field K is a K-vector space $\mathcal{L}$ equipped with an n-linear map, $[-, \ldots, -] : \mathcal{L}^{\otimes n} \to \mathcal{L}$ satisfying the Leibniz n-identity (or fundamental identity)

$$[[x_1, \ldots, x_n], y_1, \ldots, y_{n-1}] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, y_1, \ldots, y_{n-1}], x_{i+1}, \ldots, x_n].$$

A morphism of Leibniz n-algebras $\mathcal{L} \to \mathcal{L}'$ is a K-linear map that respects the n-bracket.

Example 2. (a) Let us observe that for $n = 2$ the fundamental identity is the Leibniz identity (Loday and Pirashvili, 1993) and so a Leibniz 2-algebra is simply a Leibniz algebra in the sense of Loday (1993).

(b) Clearly a Leibniz algebra $\mathfrak{g}$ is a Lie algebra if the condition $[x, x] = 0$ holds for all $x \in \mathfrak{g}$. Associated with any Leibniz algebra $\mathfrak{g}$ there is a Lie algebra $\mathfrak{g}_{\mathrm{Lie}} = \frac{\mathfrak{g}}{\mathfrak{g}^{\mathrm{ann}}}$, where $\mathfrak{g}^{\mathrm{ann}}$ is the two-sided ideal generated by $\{[x, x] |\ x \in \mathfrak{g}\}$.

(c) Similarly, for $n \geq 3$ an n-Lie algebra or an n-Nambu Lie algebra (Nambu, 1973) or Filippov algebra (Filippov, 1985) is a Leibniz n-algebra that moreover satisfies the identity

$[x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n] = 0$ if $x_i = x_j$ for $1 \leq i, j \leq n$. Such algebras have a relevant role in Nambu mechanics which is a generalization of Hamiltonian mechanics involving multiple Hamiltonians (Nambu, 1973; Takhtajan, 1994).

(d) Let V be an $(n+1)$-dimensional vector space with basis $\{e_1, e_2, \ldots, e_{n+1}\}$. Then we define $[x_1, x_2, \ldots, x_n] := \det(A)$, where A is the following matrix

$$\begin{pmatrix} e_1 & e_2 & \ldots & e_{n+1} \\ x_{11} & x_{21} & \ldots & x_{(n+1)1} \\ x_{12} & x_{22} & \ldots & x_{(n+1)2} \\ \ldots & \ldots & \ldots & \ldots \\ x_{1n} & x_{2n} & \ldots & x_{(n+1)n} \end{pmatrix}$$

and $x_i = x_{1i}e_1 + x_{2i}e_2 + \cdots + x_{(n+1)i}e_{n+1}$. Then V equipped with this n-bracket is a Leibniz n-algebra.

(e) Another big class of Leibniz 3-algebras is the so-called Lie triple systems, which were first noted by Cartan (1952) in his study on totally geodesic submanifolds and were studied from the algebraic point of view by Jacobson (1949) and Lister (1952). Let us recall that a Lie triple system is a K-vector space equipped with a 3-bracket $[-, -, -]$ that satisfies the same previous fundamental identity and the conditions $[x, y, z] + [y, z, x] + [z, x, y] = 0$ and $[x, y, y] = 0$.

We denote by ${}_n\mathbf{Lb}$ and $\mathbf{Lb}$ the categories of Leibniz n-algebras and Leibniz algebras, respectively. The Daletskii's functor (Daletskii and Takhtajan, 1997) $\mathcal{D}_n : {}_n\mathbf{Lb} \to \mathbf{Lb}$ assigns to a Leibniz n-algebra $\mathcal{L}$ the Leibniz algebra $\mathcal{D}_n(\mathcal{L}) = \mathcal{L}^{\otimes(n-1)}$ with the bracket

$$[a_1 \otimes \cdots \otimes a_{n-1}, b_1 \otimes \cdots \otimes b_{n-1}] := \sum_{i=1}^{n-1} a_1 \otimes \cdots \otimes [a_i, b_1, \ldots, b_{n-1}] \otimes \cdots \otimes a_{n-1}.$$

Conversely, the "forgetful" functor $\mathbf{U}_n : \mathbf{Lb} \to {}_n\mathbf{Lb}$ (Casas et al., 2002) means that if $\mathfrak{g}$ is a Leibniz algebra then it is also a Leibniz n-algebra with respect to the n-bracket $[-, \ldots, -] : \mathfrak{g}^{\otimes n} \to \mathfrak{g}$ given by $[x_1, \cdots, x_n] := [x_1, [x_2, \cdots [x_{n-1}, x_n]]]$.

3. Universal enveloping algebra

Definition 3. A *representation* of a Leibniz n-algebra $\mathcal{L}$ is a K-vector space M with n actions

$$[-, \overset{n}{\ldots}, -] : \mathcal{L}^{\otimes i} \otimes M \otimes \mathcal{L}^{\otimes n-i-1} \to M, \quad 0 \leq i \leq n-1,$$

satisfying the following $(2n-1)$ axioms:

R1. If $2 \leq k \leq n$,

$$\rho_k([l_1, \ldots, l_n], l_{n+1}, \ldots, l_{2n-2}) = \sum_{i=1}^{n} \rho_i(l_1, \ldots, \widehat{l_i}, \ldots, l_n) \cdot \rho_k(l_i, l_{n+1}, \ldots, l_{2n-2}),$$

R2. If $1 \leq k \leq n$,

$$[\rho_1(l_n, \ldots, l_{2n-2}), \rho_k(l_1, \ldots, l_{n-1})] = \sum_{i=1}^{n-1} \rho_k(l_1, \ldots, l_{i-1}, [l_i, l_n, \ldots, l_{2n-2}], l_{i+1}, \ldots, l_{n-1}),$$

where the multilinear applications $\rho_i : \mathcal{L}^{\otimes n-1} \to \mathrm{End}_K(M)$ are defined by

$$\rho_i(l_1,\ldots,l_{n-1})(m) := [l_1,\ldots,l_{i-1},m,l_i,\ldots,l_{n-1}], \quad 1 \le i \le n,$$

and the bracket in $\mathrm{End}_K(M)$ is the usual for associative algebras.

A particular example of representation is the case $M = \mathcal{L}$, where the applications ρ_i are the adjoint representations $\mathrm{ad}_i(l_1,\ldots,l_{n-1})(l) = [l_1,\ldots,l_{i-1},l,l_i,\ldots,l_{n-1}]$.

The notion of representation of a Leibniz n-algebra for $n = 2$ coincides with the corresponding notion given by Loday and Pirashvili (1993).

Given a Leibniz n-algebra $\mathcal{L}$, we consider n copies of the Leibniz algebra $\mathcal{L}^{\otimes(n-1)}$: one *left* copy, $(n-2)$ *middle* copies and one *right* copy, denoted by $(\mathcal{L}^{\otimes(n-1)})^l$, $(\mathcal{L}^{\otimes(n-1)})^{km}$ $(1 \le k \le n-2)$ and $(\mathcal{L}^{\otimes(n-1)})^r$, respectively.

We denote by $l_{l_1\otimes\cdots\otimes l_{n-1}}$, ${}_km_{l_1\otimes\cdots\otimes l_{n-1}}$, $1 \le k \le n-2$, $r_{l_1\otimes\cdots\otimes l_{n-1}}$ the elements of $(\mathcal{L}^{\otimes(n-1)})^l$, $(\mathcal{L}^{\otimes(n-1)})^{km}$, $1 \le k \le n-2$, and $(\mathcal{L}^{\otimes(n-1)})^r$ corresponding to $l_1 \otimes \cdots \otimes l_{n-1} \in \mathcal{L}^{\otimes(n-1)}$. We consider the tensorial algebra

$$\mathrm{T}\left((\mathcal{L}^{\otimes(n-1)})^l \oplus (\mathcal{L}^{\otimes(n-1)})^{1m} \oplus \overset{n-2}{\cdots} \oplus (\mathcal{L}^{\otimes(n-1)})^{n-2m} \oplus (\mathcal{L}^{\otimes(n-1)})^r\right)$$

and the following relations which come from R1, R2 if we adopt the following notations

$$\begin{aligned}
&\rho_1(l_1,\ldots,l_{n-1}) := l_{l_1\otimes\cdots\otimes l_{n-1}}\\
&\rho_k(l_1,\ldots,l_{n-1}) := {}_{k-1}m_{l_1\otimes\cdots\otimes l_{n-1}} \quad \text{for } 2 \le k \le n-1\\
&\rho_n(l_1,\ldots,l_{n-1}) := r_{l_1\otimes\cdots\otimes l_{n-1}},
\end{aligned}$$

where $l_1,\ldots,l_{n-1} \in \mathcal{L}$,

R1. If $2 \le k \le n-1$,

(1) ${}_{k-1}m_{[l_1,\ldots,l_n]\otimes l_{n+1}\otimes\cdots\otimes l_{2n-2}} = l_{l_2\otimes\cdots\otimes l_n} \cdot {}_{k-1}m_{l_1\otimes l_{n+1}\otimes\cdots\otimes l_{2n-2}}$
$+ \sum_{i=2}^{n-1} {}_{i-1}m_{l_1\otimes\cdots\otimes\widehat{l_i}\otimes\cdots\otimes l_n} \cdot {}_{k-1}m_{l_i\otimes l_{n+1}\otimes\cdots\otimes l_{2n-2}} + r_{l_1\otimes\cdots\otimes l_{n-1}} \cdot {}_{k-1}m_{l_n\otimes l_{n+1}\otimes\cdots\otimes l_{2n-2}}$;

if $k = n$,

(2) $r_{[l_1,\ldots,l_n]\otimes l_{n+1}\otimes\cdots\otimes l_{2n-2}} = l_{l_2\otimes\cdots\otimes l_n} \cdot r_{l_1\otimes l_{n+1}\otimes\cdots\otimes l_{2n-2}}$
$+ \sum_{i=2}^{n-1} {}_{i-1}m_{l_1\otimes\cdots\otimes\widehat{l_i}\otimes\cdots\otimes l_n} \cdot r_{l_i\otimes l_{n+1}\otimes\cdots\otimes l_{2n-2}} + r_{l_1\otimes\cdots\otimes l_{n-1}} \cdot r_{l_n\otimes l_{n+1}\otimes\cdots\otimes l_{2n-2}}$;

R2. If $k = 1$,

(3) $l_{l_n\otimes\cdots\otimes l_{2n-2}} \cdot l_{l_1\otimes\cdots\otimes l_{n-1}} - l_{l_1\otimes\cdots\otimes l_{n-1}} \cdot l_{l_n\otimes\cdots\otimes l_{2n-2}} = l_{[l_1\otimes\cdots\otimes l_{n-1},l_n\otimes\cdots\otimes l_{2n-2}]}$;

if $2 \le k \le n-1$,

(4) $l_{l_n\otimes\cdots\otimes l_{2n-2}} \cdot {}_{k-1}m_{l_1\otimes\cdots\otimes l_{n-1}} - {}_{k-1}m_{l_1\otimes\cdots\otimes l_{n-1}} \cdot l_{l_n\otimes\cdots\otimes l_{2n-2}} = {}_{k-1}m_{[l_1\otimes\cdots\otimes l_{n-1},l_n\otimes\cdots\otimes l_{2n-2}]}$;

if $k = n$,

(5) $l_{l_n\otimes\cdots\otimes l_{2n-2}} \cdot r_{l_1\otimes\cdots\otimes l_{n-1}} - r_{l_1\otimes\cdots\otimes l_{n-1}} \cdot l_{l_n\otimes\cdots\otimes l_{2n-2}} = r_{[l_1\otimes\cdots\otimes l_{n-1},l_n\otimes\cdots\otimes l_{2n-2}]}$.

Let us observe that from the relations R1(2) and R2(5) we derive the following relation

(1)′ $\sum_{i=2}^{n-1} {}_{i-1}m_{l_1\otimes\cdots\otimes\widehat{l_i}\otimes\cdots\otimes l_n} \cdot r_{l_i\otimes l_{n+1}\otimes\ldots l_{2n-2}} + r_{l_1\otimes\cdots\otimes l_{n-1}} \cdot r_{l_n\otimes l_{n+1}\otimes\cdots\otimes l_{2n-2}}$
$+ r_{l_1\otimes l_{n+1}\otimes\cdots\otimes l_{2n-2}} \cdot l_{l_2\otimes\cdots\otimes l_n} + \sum_{i=n+1}^{2n-2} r_{l_1\otimes\underbrace{l_{n+1}\otimes\cdots\otimes l_{i-1}}_{i-n-1}\otimes[l_i,l_2,\ldots,l_n]\otimes l_{i+1}\cdots\otimes l_{2n-2}} = 0$

and from the relations R1(1) and R2(4) we deduce the following relation for all $2 \le k \le n-1$

(2)′ $r_{l_1\otimes\cdots\otimes l_{n-1}} \cdot {}_{k-1}m_{l_n\otimes l_{n+1}\otimes\cdots\otimes l_{2n-2}} + \sum_{i=2}^{n-1} {}_{i-1}m_{l_1\otimes\cdots\otimes\widehat{l_i}\otimes\cdots\otimes l_n} \cdot {}_{k-1}m_{l_i\otimes l_{n+1}\otimes\cdots\otimes l_{2n-2}}$
$+ {}_{k-1}m_{l_1\otimes l_{n+1}\otimes\cdots\otimes l_{2n-2}} \cdot l_{l_2\otimes\cdots\otimes l_n} + \sum_{i=n+1}^{2n-2} {}_{k-1}m_{l_1\otimes\underbrace{l_{n+1}\otimes\cdots\otimes l_{i-1}}_{i-n-1}\otimes[l_i,l_2,\ldots,l_n]\otimes l_{i+1}\cdots\otimes l_{2n-2}}$
$= 0.$

The cancelation of R2(3) on itself induces the following relation
(3)′ $l_{[l_1\otimes\cdots\otimes l_{n-1}, l_n\otimes\cdots\otimes l_{2n-2}]} + l_{[l_n\otimes\cdots\otimes l_{2n-2}, l_1\otimes\cdots\otimes l_{n-1}]} = 0$.

Definition 4. The *universal enveloping algebra* of the Leibniz n-algebra $\mathcal{L}$ is the unitary associative algebra

$$\mathrm{U_nL}(\mathcal{L}) := \mathrm{T}\left((\mathcal{L}^{\otimes(n-1)})^l \oplus (\mathcal{L}^{\otimes(n-1)})^{1m} \oplus \overset{n-2}{\cdots} \oplus (\mathcal{L}^{\otimes(n-1)})^{n-2m} \oplus (\mathcal{L}^{\otimes(n-1)})^r\right)/I,$$

where I is the n-sided ideal corresponding to the relations R1(1), (2) and R2(3), (4), (5).

Proposition 5. *The category of representations of the Leibniz n-algebra $\mathcal{L}$ is equivalent to the category of right modules over* $\mathrm{U_nL}(\mathcal{L})$.

Proof. Let M be a representation of $\mathcal{L}$. We define a right action from $\mathrm{U_nL}(\mathcal{L})$ on the K-vector space M as follows. Firstly $(\mathcal{L}^{\otimes(n-1)})^l$, $(\mathcal{L}^{\otimes(n-1)})^{km}$, $1 \le k \le n-2$, $(\mathcal{L}^{\otimes(n-1)})^r$ act on M by

$$\begin{aligned}
&m \cdot l_{l_1\otimes\cdots\otimes l_{n-1}} = [m, l_1, \ldots, l_{n-1}],\\
&m \cdot {}_i m_{l_1\otimes\cdots\otimes l_{n-1}} = [l_1, \ldots, l_i, m, l_{i+1}, \ldots, l_{n-1}], \quad 1 \le i \le n-2,\\
&m \cdot r_{l_1\otimes\cdots\otimes l_{n-1}} = [l_1, \ldots, l_{n-1}, m];
\end{aligned}$$

then we extend these actions to an action of

$$\mathrm{T}\left((\mathcal{L}^{\otimes(n-1)})^l \oplus (\mathcal{L}^{\otimes(n-1)})^{1m} \oplus \overset{n-2}{\cdots} \oplus (\mathcal{L}^{\otimes(n-1)})^{n-2m} \oplus (\mathcal{L}^{\otimes(n-1)})^r\right)$$

by composition and linearity.

The axioms R1 and R2 of the representation imply that the relations R1(1), (2) and R2(3), (4), (5) act trivially. So M is endowed with a structure of $\mathrm{U_nL}(\mathcal{L})$-module.

Conversely, we start with a $\mathrm{U_nL}(\mathcal{L})$-module. The restriction of the actions to $(\mathcal{L}^{\otimes(n-1)})^l$, $(\mathcal{L}^{\otimes(n-1)})^{1m}, \overset{n-2}{\cdots}, (\mathcal{L}^{\otimes(n-1)})^{n-2m}$, $(\mathcal{L}^{\otimes(n-1)})^r$ provides n actions of $\mathcal{L}^{\otimes(n-1)}$ which makes M a representation of $\mathcal{L}$. $\square$

Remark 6. In case $n = 2$, that is, for Leibniz algebras, we recover the construction of the universal enveloping algebra of a Leibniz algebra and Proposition 5 reproduces Loday and Pirashvili (1993, (2.3) Theorem).

In case $n = 3$, we recover the construction of the universal enveloping algebra of a Leibniz 3-algebra and Proposition 5 reproduces Casas (2006, Theorem 4.3).

Thanks to relation R2(3) we have that the subalgebra spanned by the elements $l_{l_1\otimes\cdots\otimes l_{n-1}}$, $l_1 \otimes \cdots \otimes l_{n-1} \in \mathcal{L}^{\otimes n-1}$, is isomorphic to $\mathrm{U}((\mathcal{L}^{\otimes n-1})_{\mathrm{Lie}})$.

Definition 7. Let $\mathcal{L}$ be a Leibniz n-algebra and A an associative algebra. An *n-homomorphism* from $\mathcal{L}$ to A consists in an n-tuple of K-linear maps $(\varphi_1, \ldots, \varphi_n)$, $\varphi_i : \mathcal{L}^{\otimes n-1} \to A$, $1 \le i \le n$, satisfying the following relations:

(a) if $2 \le i \le n$,

$$\begin{aligned}
\varphi_i([l_1, \ldots, l_n] \otimes l_{n+1} \otimes \cdots \otimes l_{2n-2}) &= \varphi_1(l_2 \otimes \cdots \otimes l_n) \cdot \varphi_i(l_1 \otimes l_{n+1} \otimes \cdots \otimes l_{2n-2})\\
&+ \sum_{j=2}^{n-1} \varphi_j(l_1 \otimes \cdots \otimes \widehat{l_j} \otimes \cdots \otimes l_n) \cdot \varphi_i(l_j \otimes l_{n+1} \otimes \cdots \otimes l_{2n-2})\\
&+ \varphi_n(l_1 \otimes \cdots \otimes l_{n-1}) \cdot \varphi_i(l_n \otimes l_{n+1} \otimes \cdots \otimes l_{2n-2}),
\end{aligned}$$

(b) if $1 \leq i \leq n$,

$$\varphi_i[l_1 \otimes \cdots \otimes l_{n-1}, l_n \otimes \cdots \otimes l_{2n-2}] = \varphi_1(l_n \otimes \cdots \otimes l_{2n-2}) \cdot \varphi_i(l_1 \otimes \cdots \otimes l_{n-1}) - \varphi_i(l_1 \otimes \cdots \otimes l_{n-1}) \cdot \varphi_1(l_n \otimes \cdots \otimes l_{2n-2}).$$

Remark 8. In case $n = 2$ we recover the definition of bihomomorphism (Aymon, 1997) and in case $n = 3$ we recover the definition of trihomomorphism (Casas, 2006).

Given a Leibniz n-algebra $\mathcal{L}$ there exists a canonical n-homomorphism $(\varphi_1, \ldots, \varphi_n)$ from $\mathcal{L}$ to $\mathrm{U_nL}(\mathcal{L})$ given by

$$\varphi_i(l_1 \otimes \cdots \otimes l_{n-1}) = \begin{cases} l_{l_1 \otimes \cdots \otimes l_{n-1}} & \text{for } i = 1, \\ {}_{i-1}m_{l_1 \otimes \cdots \otimes l_{n-1}} & \text{for } 2 \leq i \leq n-1, \\ r_{l_1 \otimes \cdots \otimes l_{n-1}} & \text{for } i = n. \end{cases}$$

Proposition 9 (*Universal Property*). *The canonical n-homomorphism $(\varphi_1, \ldots, \varphi_n) : \mathcal{L}^{\otimes n-1} \to \mathrm{U_nL}(\mathcal{L})$ is universal for the n-homomorphisms of $\mathcal{L}$, that is,* ${}_{\mathrm{n}}\mathrm{Hom}(\mathcal{L}, \mathrm{A}) \cong \mathrm{Ass}(\mathrm{U_nL}(\mathcal{L}), \mathrm{A})$.

Proof. Let $(\theta_1, \ldots, \theta_n)$ be an n-homomorphism from $\mathcal{L}$ to A. We define a K-linear homomorphism $(\mathcal{L}^{\otimes(n-1)})^l \oplus (\mathcal{L}^{\otimes(n-1)})^{{}_1m} \oplus \overset{n-2}{\cdots} \oplus (\mathcal{L}^{\otimes(n-1)})^{{}_{n-2}m} \oplus (\mathcal{L}^{\otimes(n-1)})^r \to \mathrm{A}$ by $l_{l_1 \otimes \cdots \otimes l_{n-1}} \mapsto \theta_1(l_1 \otimes \cdots \otimes l_{n-1})$, ${}_km_{l_1 \otimes \cdots \otimes l_{n-1}} \mapsto \theta_{k+1}(l_1 \otimes \cdots \otimes l_{n-1})$, $1 \leq k \leq n-2$, $r_{l_1 \otimes \cdots \otimes l_{n-1}} \mapsto \theta_n(l_1 \otimes \cdots \otimes l_{n-1})$ which extends to $T((\mathcal{L}^{\otimes(n-1)})^l \oplus (\mathcal{L}^{\otimes(n-1)})^{{}_1m} \oplus \overset{n-2}{\cdots} \oplus (\mathcal{L}^{\otimes(n-1)})^{{}_{n-2}m} \oplus (\mathcal{L}^{\otimes(n-1)})^r)$ and vanishes on I, so it induces a homomorphism of associative algebras $\mathrm{U_nL}(\mathcal{L}) \to \mathrm{A}$.

Conversely, given a homomorphism of associative algebras $f : \mathrm{U_nL}(\mathcal{L}) \to \mathrm{A}$, the n-tuple $(f \circ \varphi_1, \ldots, f \circ \varphi_n)$ is an n-homomorphism of $\mathcal{L}$. Moreover, both processes are inverse. □

4. PBW theorem for Leibniz *n*-algebras

Loday and Pirashvili (1993) show a Poincaré–Birkhoff–Witt theorem for any Leibniz algebra. Here, we give a proof of a PBW theorem for finite-dimensional Leibniz n-algebras using the theory of Gröbner bases.

Let $\mathcal{L}$ be a finite-dimensional Leibniz n-algebra with a K-basis $\{e_1, \ldots, e_d\}$.

Identifying

$$\mathrm{T}\left((\mathcal{L}^{\otimes(n-1)})^l \oplus (\mathcal{L}^{\otimes(n-1)})^{{}_1m} \oplus \overset{n-2}{\cdots} \oplus (\mathcal{L}^{\otimes(n-1)})^{{}_{n-2}m} \oplus (\mathcal{L}^{\otimes(n-1)})^r\right)$$

with $K<X_{s_1,\ldots,s_{n-1}}, {}_kY_{s_1,\ldots,s_{n-1}}, Z_{s_1,\ldots,s_{n-1}}>$, $s_1, \ldots, s_{n-1} \in \{1, \ldots, d\}$, $1 \leq k \leq n-2$, via the morphism $\Phi(l_{e_{s_1} \otimes \cdots \otimes e_{s_{n-1}}}) = x_{s_1,\ldots,s_{n-1}}$, $\Phi({}_km_{e_{s_1} \otimes \cdots \otimes e_{s_{n-1}}}) = {}_ky_{s_1,\ldots,s_{n-1}}$, $\Phi(r_{e_{s_1} \otimes \cdots \otimes e_{s_{n-1}}}) = z_{s_1,\ldots,s_{n-1}}$, the relations R1(1), (2) and R2(3), (4), (5) are translated into

(1) $\Phi({}_{k-1}m_{[e_{s_1},\ldots,e_{s_n}] \otimes e_{s_{n+1}} \otimes \cdots \otimes e_{s_{2n-2}}}) = x_{s_2,\ldots,s_n} \cdot {}_{k-1}y_{s_1,s_{n+1},\ldots,s_{2n-2}} + \sum_{i=2}^{n-1} {}_{i-1}y_{s_1,\ldots,\widehat{s_i},\ldots,s_n} \cdot {}_{k-1}y_{s_i,s_{n+1},\ldots,s_{2n-2}} + z_{s_1,\ldots,s_{n-1}} \cdot {}_{k-1}y_{s_n,s_{n+1},\ldots,s_{2n-2}}$

(2) $\Phi(r_{[e_{s_1},\ldots,e_{s_n}] \otimes e_{s_{n+1}} \otimes \cdots \otimes e_{s_{2n-2}}}) = x_{s_2,\ldots,s_n} \cdot z_{s_1,s_{n+1},\ldots,s_{2n-2}} + \sum_{i=2}^{n-1} {}_{i-1}y_{s_1,\ldots,\widehat{s_i},\ldots,s_n} \cdot z_{s_i,s_{n+1},\ldots,s_{2n-2}} + z_{s_1,\ldots,s_{n-1}} \cdot z_{s_n,s_{n+1},\ldots,s_{2n-2}}$

(3) $x_{s_n,s_{n+1},\ldots,s_{2n-2}} \cdot x_{s_1,\ldots,s_{n-1}} - x_{s_1,\ldots,s_{n-1}} \cdot x_{s_n,s_{n+1},\ldots,s_{2n-2}} = \Phi(l_{[e_{s_1} \otimes \cdots \otimes e_{s_{n-1}}, e_{s_n} \otimes \cdots \otimes e_{s_{2n-2}}]})$

(4) $x_{s_n,\ldots,s_{2n-2}} \cdot {}_{k-1}y_{s_1,\ldots,s_{n-1}} - {}_{k-1}y_{s_1,\ldots,s_{n-1}} \cdot x_{s_n,\ldots,s_{2n-2}} = \Phi({}_{k-1}m_{[e_{s_1} \otimes \cdots \otimes e_{s_{n-1}}, e_{s_n} \otimes \cdots \otimes e_{s_{2n-2}}]})$

(5) $x_{s_n,\ldots,s_{2n-2}} \cdot z_{s_1,\ldots,s_{n-1}} - z_{s_1,\ldots,s_{n-1}} \cdot x_{s_n,\ldots,s_{2n-2}} = \Phi(r_{[e_{s_1} \otimes \cdots \otimes e_{s_{n-1}}, e_{s_n} \otimes \cdots \otimes e_{s_{2n-2}}]})$

Thus, we can use the theory of Gröbner bases on $K{<}X_{s_1,\ldots,s_{n-1}},\ {}_kY_{s_1,\ldots,s_{n-1}}, Z_{s_1,\ldots,s_{n-1}}{>}$ (Green, 1994; Mora, 1994) to obtain results in the universal enveloping algebra $\mathrm{U_nL}(\mathcal{L})$. We will obtain a proof of a PBW theorem for Leibniz n-algebras calculating a Gröbner basis of the ideal $\Phi(I) \subset K{<}X_{s_1,\ldots,s_{n-1}},\ {}_kY_{s_1,\ldots,s_{n-1}}, Z_{s_1,\ldots,s_{n-1}}{>}$ generated by the relations (1)–(5).

We fix the degree lexicographical ordering on $K{<}X_{s_1,\ldots,s_{n-1}},\ {}_kY_{s_1,\ldots,s_{n-1}}, Z_{s_1,\ldots,s_{n-1}}{>}$ with $Z_{s_1,\ldots,s_{n-1}} > {}_kY_{s_1,\ldots,s_{n-1}} > X_{s_1,\ldots,s_{n-1}}$ and each variable is ordered by the subindex with respect to the left degree lexicographical ordering.

Let $\{g_1, \ldots, g_p\}$ be a K-basis of $(\mathcal{L}^{\otimes(n-1)})^{\mathrm{ann}}$. We adopt the following notation. If $X = \{x_{i_1,\ldots,i_{n-1}}\}_{i_1,\ldots,i_{n-1}\in\{1,\ldots,d\}}$ we will denote by $x^{\#a}_{i_1,\ldots,i_{n-1}}$ the element that it is a places to the right of $x_{i_1,\ldots,i_{n-1}}$ in X if $a \geq 0$, and a places to the left of $x_{i_1,\ldots,i_{n-1}}$ in X if $a < 0$ and by $\alpha_1, \ldots, \alpha_{n-1} \in \{1, \ldots, d\}$ the indexes such that $x^{\#(p-1)}_{\alpha_1,\ldots,\alpha_{n-1}} = x_{dd\ldots d}$. So, we have $X = X_1 \bigsqcup X_2$ where $X_1 = \{x_{11\ldots1}, \ldots, x^{\#(-1)}_{\alpha_1,\ldots,\alpha_{n-1}}\}$ and $X_2 = \{x^{\#(0)}_{\alpha_1,\ldots,\alpha_{n-1}}, \ldots, x^{\#(p-1)}_{\alpha_1,\ldots,\alpha_{n-1}}\}$.

Moreover we can suppose (to simplify the notation) that the basis has the form

$$\begin{aligned} g_1 &= x^{\#(0)}_{\alpha_1,\ldots,\alpha_{n-1}} + f_1(X_1)\\ g_2 &= x^{\#(1)}_{\alpha_1,\ldots,\alpha_{n-1}} + f_2(X_1)\\ &\ldots\\ g_p &= x^{\#(p-1)}_{\alpha_1,\ldots,\alpha_{n-1}} + f_p(X_1). \end{aligned}$$

The proof of this theorem is divided into three parts. In the first one, we proceed to identify the two-sided ideal $(\mathcal{L}^{\otimes(n-1)})^{\mathrm{ann}}$ generated by $\{[x, x] \mid x \in \mathcal{L}^{\otimes(n-1)}\}$, since this ideal plays an outstanding role in the demonstration. Secondly, we obtain a minimal set G from the generators of $\Phi(I)$; and, in the third part, we verify that G is a Gröbner basis of $\Phi(I)$. We have divided the proof into a sequence of lemmas.

Lemma 10. *The ideal $(\mathcal{L}^{\otimes(n-1)})^{\mathrm{ann}}$ is generated by* $\{[e_{i_1} \otimes \cdots \otimes e_{i_{n-1}}, e_{i_1} \otimes \cdots \otimes e_{i_{n-1}}]\}_{i_1,\ldots,i_{n-1}\in\{1,\ldots,d\}} \cup \{[e_{i_1} \otimes \cdots \otimes e_{i_{n-1}}, e_{j_1} \otimes \cdots \otimes e_{j_{n-1}}] + [e_{j_1} \otimes \cdots \otimes e_{j_{n-1}}, e_{i_1} \otimes \cdots \otimes e_{i_{n-1}}]\}_{i_1,\ldots,i_{n-1},j_1,\ldots,j_{n-1}\in\{1,\ldots,d\},(i_1,\ldots,i_{n-1})\neq(j_1,\ldots,j_{n-1})}$.

Proof. Since $\mathcal{L}^{\otimes(n-1)}$ is a Leibniz algebra, the proof is analogous to Insua and Ladra (submitted for publication, Theorem 5). □

Lemma 11. *The ideal generated by the relations* (3) $\{x_{s_n,s_{n+1},\ldots,s_{2n-2}} \cdot x_{s_1,\ldots,s_{n-1}} - x_{s_1,\ldots,s_{n-1}} \cdot x_{s_n,s_{n+1},\ldots,s_{2n-2}} - \Phi(l_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}, e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]})\}$ *is the same as the ideal generated by* $C = \{x_{s_n,s_{n+1},\ldots,s_{2n-2}} \cdot x_{s_1,\ldots,s_{n-1}} - x_{s_1,\ldots,s_{n-1}} \cdot x_{s_n,s_{n+1},\ldots,s_{2n-2}} - \Phi(l_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}, e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]})\} \cup \{g_1, \ldots, g_p\}$ *with* $s_1, \ldots, s_{2n-2} \in \{1, \ldots, d\}$ *and* $(s_1, \ldots, s_{n-1}) < (s_n, \ldots, s_{2n-2})$.

Proof. Straightforward. □

Lemma 12. *The ideal generated by C is the same as the ideal generated by* $D = \{x_{s_n,s_{n+1},\ldots,s_{2n-2}} \cdot x_{s_1,\ldots,s_{n-1}} - x_{s_1,\ldots,s_{n-1}} \cdot x_{s_n,s_{n+1},\ldots,s_{2n-2}} - \Phi(l_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}, e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]})\} \cup \{g_1, \ldots, g_p\}$ *with* $s_1, \ldots, s_{2n-2} \in \{1, \ldots, d\}$, $(s_1, \ldots, s_{n-1}) < (s_n, \ldots, s_{2n-2})$ *and* $x_{s_n,s_{n+1},\ldots,s_{2n-2}} \notin X_2$.

Proof. Let $x_{j_n,j_{n+1},\ldots,j_{2n-2}} \cdot x_{j_1,\ldots,j_{n-1}} - x_{j_1,\ldots,j_{n-1}} \cdot x_{j_n,j_{n+1},\ldots,j_{2n-2}} - \Phi(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, e_{j_n}\otimes\cdots\otimes e_{j_{2n-2}}]})$ such that $x_{j_n,j_{n+1},\ldots,j_{2n-2}} \in X_2$ and $s \in \mathbb{N}$ such that $x^{\#(s-1)}_{\alpha_1,\ldots,\alpha_{n-1}} = x_{j_n,j_{n+1},\ldots,j_{2n-2}}$.

$x_{j_n,j_{n+1},\ldots,j_{2n-2}} \cdot x_{j_1,\ldots,j_{n-1}} - x_{j_1,\ldots,j_{n-1}} \cdot x_{j_n,j_{n+1},\ldots,j_{2n-2}} - \Phi(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, e_{j_n}\otimes\cdots\otimes e_{j_{2n-2}}]})$

$\rightarrow_{\{-g_s \cdot x_{j_1,\ldots,j_{n-1}}, x_{j_1,\ldots,j_{n-1}} \cdot g_s\}} -\Phi(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, e_{j_n}\otimes\cdots\otimes e_{j_{2n-2}}]}) - f_s(X_1)\cdot x_{j_1,\ldots,j_{n-1}} + x_{j_1,\ldots,j_{n-1}} \cdot f_s(X_1) = -(a^s_{11\ldots1}\cdot x_{11\ldots1} + \cdots + \widehat{a}^s_{\alpha_1,\ldots,\alpha_{n-1}}\cdot \widehat{x}_{\alpha_1,\ldots,\alpha_{n-1}})\cdot x_{j_1,\ldots,j_{n-1}} + x_{j_1,\ldots,j_{n-1}}\cdot (a^s_{11\ldots1}\cdot x_{11\ldots1} + \cdots + \widehat{a}^s_{\alpha_1,\ldots,\alpha_{n-1}}\cdot \widehat{x}_{\alpha_1,\ldots,\alpha_{n-1}}) - \Phi(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, e_{j_n}\otimes\cdots\otimes e_{j_{2n-2}}]}) = a^s_{11\ldots1}\cdot(x_{j_1,\ldots,j_{n-1}}\cdot x_{11\ldots1} - x_{11\ldots1}\cdot x_{j_1,\ldots,j_{n-1}}) + \cdots + \widehat{a}^s_{\alpha_1,\ldots,\alpha_{n-1}}\cdot(\widehat{x}_{j_1,\ldots,j_{n-1}}\cdot\widehat{x}_{\alpha_1,\ldots,\alpha_{n-1}} - \widehat{x}_{\alpha_1,\ldots,\alpha_{n-1}}\cdot\widehat{x}_{j_1,\ldots,j_{n-1}}) - \Phi(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, e_{j_n}\otimes\cdots\otimes e_{j_{2n-2}}]})$.

So two different cases are possible: $x_{j_1,\ldots,j_{n-1}} \in X_2$ or $x_{j_1,\ldots,j_{n-1}} \notin X_2$.

(I) $x_{j_1,\ldots,j_{n-1}} \in X_2$.

Let be $g_r = \underbrace{x^{\#(r-1)}_{\alpha_1,\ldots,\alpha_{n-1}}}_{x_{j_1,\ldots,j_{n-1}}} + f_r(X_1)$.

For every $x_{t_1,\ldots,t_{n-1}} \in X_1$ it is verified

$x_{j_1,\ldots,j_{n-1}}\cdot x_{t_1,\ldots,t_{n-1}} - x_{t_1,\ldots,t_{n-1}}\cdot x_{j_1,\ldots,j_{n-1}} \rightarrow_{\{-g_r\cdot x_{t_1,\ldots,t_{n-1}}, x_{t_1,\ldots,t_{n-1}}\cdot g_r\}} -f_r(X_1)\cdot x_{t_1,\ldots,t_{n-1}} + x_{t_1,\ldots,t_{n-1}}\cdot f_r(X_1) = a^r_{11\ldots1}\cdot(-x_{11\ldots1}\cdot x_{t_1,\ldots,t_{n-1}} + x_{t_1,\ldots,t_{n-1}}\cdot x_{11\ldots1}) + \cdots + a^r_{t_1,\ldots,t_{n-1}}\cdot \underbrace{(-x_{t_1,\ldots,t_{n-1}}\cdot x_{t_1,\ldots,t_{n-1}} + x_{t_1,\ldots,t_{n-1}}\cdot x_{t_1,\ldots,t_{n-1}})}_{0} + \cdots + \widehat{a}^r_{\alpha_1,\ldots,\alpha_{n-1}}\cdot(-\widehat{x}_{\alpha_1,\ldots,\alpha_{n-1}}\cdot \widehat{x}_{t_1,\ldots,t_{n-1}} + \widehat{x}_{t_1,\ldots,t_{n-1}}\cdot\widehat{x}_{\alpha_1,\ldots,\alpha_{n-1}}) \rightarrow_D a^r_{11\ldots1}\cdot\Phi(l_{[e_1\otimes\cdots\otimes e_1, e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}]}) + \cdots + 0 - \cdots - \widehat{a}^r_{\alpha_1,\ldots,\alpha_{n-1}}\cdot\widehat{\Phi}(l_{[e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_{n-1}}]}) \rightarrow_D -\Phi(l_{[e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}, -e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}+g_r]}) = \Phi(l_{[e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}, e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}]}) - \Phi(l_{[e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}, g_r]}) = \Phi(l_{[e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}, e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}]})$ because $[e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}, g_r] = 0$ by the fact that $g_r \in (\mathcal{L}^{\otimes(n-1)})^{\mathrm{ann}}$.

Then the initial expression reduces to

$-\Phi(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, e_{j_n}\otimes\cdots\otimes e_{j_{2n-2}}]}) - a^s_{11\ldots1}\cdot\Phi(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, e_1\otimes\cdots\otimes e_1]}) - \cdots - \widehat{a}^s_{\alpha_1,\ldots,\alpha_{n-1}}\cdot\widehat{\Phi}(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_{n-1}}]}) = -\Phi(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, g_s]}) = 0$ because $g_r \in (\mathcal{L}^{\otimes(n-1)})^{\mathrm{ann}}$.

(II) $x_{j_1,\ldots,j_{n-1}} \in X_1$.

The initial expression reduces to

$-\Phi(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, e_{j_n}\otimes\cdots\otimes e_{j_{2n-2}}]}) + a^s_{11\ldots1}\cdot\Phi(l_{[e_1\otimes\cdots\otimes e_1, e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}]}) + \cdots + a^s_{j_n,\ldots,j_{2n-2}}\cdot 0 - \cdots - \widehat{a}^s_{\alpha_1,\ldots,\alpha_{n-1}}\cdot\widehat{\Phi}(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_{n-1}}]}) \rightarrow_D -\Phi(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, e_{j_n}\otimes\cdots\otimes e_{j_{2n-2}}]}) - a^s_{11\ldots1}\cdot\Phi(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, e_1\otimes\cdots\otimes e_1]}) - \cdots - \widehat{a}^s_{\alpha_1,\ldots,\alpha_{n-1}}\cdot\widehat{\Phi}(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_{n-1}}]}) = -\Phi(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}}, g_s]}) = 0$. □

Lemma 13. *The ideal generated by the relations* (3), (4) *and* (5) *agrees with the ideal generated by* $E = D \cup \{x_{s_n,\ldots,s_{2n-2}}\cdot {}_{k-1}y_{s_1,\ldots,s_{n-1}} - {}_{k-1}y_{s_1,\ldots,s_{n-1}}\cdot x_{s_n,\ldots,s_{2n-2}} - \Phi({}_{k-1}m_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}, e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]})\} \cup \{x_{s_n,\ldots,s_{2n-2}}\cdot z_{s_1,\ldots,s_{n-1}} - z_{s_1,\ldots,s_{n-1}}\cdot x_{s_n,\ldots,s_{2n-2}} - \Phi(r_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}, e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]})\}$ *with* $s_1,\ldots,s_{2n-2} \in \{1,\ldots,d\}$, $x_{s_n,s_{n+1},\ldots,s_{2n-2}} \notin X_2$ *and* $k \in \{2,\ldots,n-1\}$.

Proof. Let $s_1,\ldots,s_{2n-2} \in \{1,\ldots,d\}$ such that $x_{s_n,\ldots,s_{2n-2}} \in X_2$ and $r \in \{1,\ldots,p\}$ such that $g_r = x_{s_n,\ldots,s_{2n-2}} + f_r(X_1)$.

$x_{s_n,\ldots,s_{2n-2}}\cdot_{k-1} y_{s_1,\ldots,s_{n-1}} -_{k-1} y_{s_1,\ldots,s_{n-1}}\cdot x_{s_n,\ldots,s_{2n-2}} - \Phi({}_{k-1}m_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}, e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]}) \rightarrow_{\{{}_{k-1}y_{s_1,\ldots,s_{n-1}}\cdot g_r, g_r\cdot{}_{k-1}y_{s_1,\ldots,s_{n-1}}\}} -\Phi({}_{k-1}m_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}, e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]}) + {}_{k-1}y_{s_1,\ldots,s_{n-1}}\cdot f_r(X_1) - f_r(X_1)\cdot_{k-1} y_{s_1,\ldots,s_{n-1}} = -\Phi({}_{k-1}m_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}, e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]}) + a^r_{11\ldots1}\cdot({}_{k-1}y_{s_1,\ldots,s_{n-1}}\cdot x_{11\ldots1} - x_{11\ldots1}\cdot_{k-1} y_{s_1,\ldots,s_{n-1}}) + \cdots + \widehat{a}^r_{\alpha_1,\ldots,\alpha_{n-1}}\cdot({}_{k-1}\widehat{y}_{s_1,\ldots,s_{n-1}}\cdot\widehat{x}_{\alpha_1,\ldots,\alpha_{n-1}} - \widehat{x}_{\alpha_1,\ldots,\alpha_{n-1}}\cdot{}_{k-1}\widehat{y}_{s_1,\ldots,s_{n-1}}) \rightarrow_E -\Phi({}_{k-1}m_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}, e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]}) - a^r_{11\ldots1}\cdot\Phi({}_{k-1}m_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}, e_{s_1}\otimes\cdots\otimes e_1]}) - \cdots - \widehat{a}^r_{\alpha_1,\ldots,\alpha_{n-1}}\cdot\widehat{\Phi}({}_{k-1}m_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_{n-1}}]}) = -\Phi({}_{k-1}m_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}, g_s]}) = 0$.

If we repeat again the previous process for the another relation we will obtain $x_{s_n,\ldots,s_{2n-2}}\cdot z_{s_1,\ldots,s_{n-1}} - z_{s_1,\ldots,s_{n-1}}\cdot x_{s_n,\ldots,s_{2n-2}} - \Phi(r_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}, e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]}) \rightarrow_E 0$. □

Proposition 14. *A minimal Gröbner basis of the ideal generated by the relations* (1)–(5) *is* $G = E \cup \{x_{s_2,\dots,s_n} \cdot {}_{k-1}y_{s_1,s_{n+1},\dots,s_{2n-2}} + \sum_{i=2}^{n-1} {}_{i-1}y_{s_1,\dots,\widehat{s_i},\dots,s_n} \cdot {}_{k-1}y_{s_i,s_{n+1},\dots,s_{2n-2}} + z_{s_1,\dots,s_{n-1}} \cdot {}_{k-1}y_{s_n,s_{n+1},\dots,s_{2n-2}} - \Phi({}_{k-1}m_{[e_{s_1},\dots,e_{s_n}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}})\} \cup \{x_{s_2,\dots,s_n} \cdot z_{s_1,s_{n+1},\dots,s_{2n-2}} + \sum_{i=2}^{n-1} {}_{i-1}y_{s_1,\dots,\widehat{s_i},\dots,s_n} \cdot z_{s_i,s_{n+1},\dots,s_{2n-2}} + z_{s_1,\dots,s_{n-1}} \cdot z_{s_n,s_{n+1},\dots,s_{2n-2}} - \Phi(r_{[e_{s_1},\dots,e_{s_n}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}})\}$ *with respect to the degree lexicographical ordering on* $K<X_{s_1,\dots,s_{n-1}}, {}_kY_{s_1,\dots,s_{n-1}}, Z_{s_1,\dots,s_{n-1}}>$ *with* $Z_{s_1,\dots,s_{n-1}} > {}_kY_{s_1,\dots,s_{n-1}} > X_{s_1,\dots,s_{n-1}}$.

Proof. We will prove that all overlap relations of elements of G reduce to 0 modulo G. We will only show in detail two typical cases (cases 1 and 4) of the seven possible ones to give an idea of the proof. The other ones can be proved analogously using the overlap relations given here.

Case 1. Denote by $g_{s_n,\dots,s_{2n-2},s_1,\dots,s_{n-1}}$ the element $x_{s_n,s_{n+1},\dots,s_{2n-2}} \cdot x_{s_1,\dots,s_{n-1}} - x_{s_1,\dots,s_{n-1}} \cdot x_{s_n,s_{n+1},\dots,s_{2n-2}} - \Phi(l_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}},e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]})$.

If $x_{s_n,\dots,s_{2n-2}} > x_{s_1,\dots,s_{n-1}} > x_{t_1,\dots,t_{n-1}}$, then

OR1 $:= g_{s_n,\dots,s_{2n-2},s_1,\dots,s_{n-1}} \cdot x_{t_1,\dots,t_{n-1}} - x_{s_n,\dots,s_{2n-2}} \cdot g_{s_1,\dots,s_{n-1},t_1,\dots,t_{n-1}} = -x_{s_1,\dots,s_{n-1}} \cdot x_{s_n,\dots,s_{2n-2}} \cdot x_{t_1,\dots,t_{n-1}} - \Phi(l_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}},e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]}) \cdot x_{t_1,\dots,t_{n-1}} + x_{s_n,\dots,s_{2n-2}} \cdot x_{t_1,\dots,t_{n-1}} \cdot x_{s_1,\dots,s_{n-1}} + x_{s_n,\dots,s_{2n-2}} \cdot \Phi(l_{[e_{t_1}\otimes\cdots\otimes e_{t_{n-1}},e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}]}) \to_G -\Phi(l_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}},e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]}) \cdot x_{t_1,\dots,t_{n-1}} + x_{s_n,\dots,s_{2n-2}} \cdot \Phi(l_{[e_{t_1}\otimes\cdots\otimes e_{t_{n-1}},e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}]}) - x_{s_1,\dots,s_{n-1}} \cdot \Phi(l_{[e_{t_1}\otimes\cdots\otimes e_{t_{n-1}},e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]}) + \Phi(l_{[e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}},e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}]}) \cdot x_{s_1,\dots,s_{n-1}} - \Phi(l_{[e_{t_1}\otimes\cdots\otimes e_{t_{n-1}},e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}]}) \cdot x_{s_n,\dots,s_{2n-2}} + x_{t_1,\dots,t_{n-1}} \cdot \Phi(l_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}},e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]})$.

Since $\Phi(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}},e_{r_1}\otimes\cdots\otimes e_{r_{n-1}}]}) \cdot x_{c_1,\dots,c_{n-1}} - x_{c_1,\dots,c_{n-1}} \cdot \Phi(l_{[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}},e_{r_1}\otimes\cdots\otimes e_{r_{n-1}}]}) \to_G \Phi(l_{[[e_{j_1}\otimes\cdots\otimes e_{j_{n-1}},e_{r_1}\otimes\cdots\otimes e_{r_{n-1}}],e_{c_1}\otimes\cdots\otimes e_{c_{n-1}}]})$ then the overlap relation OR1 reduces to $\Phi(l_{[e_{t_1}\otimes\cdots\otimes e_{t_{n-1}},[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}},e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]]}) - \Phi(l_{[[e_{t_1}\otimes\cdots\otimes e_{t_{n-1}},e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}],e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]}) + \Phi(l_{[[e_{t_1}\otimes\cdots\otimes e_{t_{n-1}},e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}],e_{s_1}\otimes\cdots\otimes e_{s_{n-1}}]}) = 0$ modulo G by the Leibniz identity of $\mathcal{L}^{\otimes(n-1)}$.

Case 2. Denote by $h^{k-1}_{y_{s_1,\dots,s_{n-1},s_n,\dots,s_{2n-2}}}$ the element $-x_{s_n,\dots,s_{2n-2}} \cdot {}_{k-1}y_{s_1,\dots,s_{n-1}} + {}_{k-1}y_{s_1,\dots,s_{n-1}} \cdot x_{s_n,\dots,s_{2n-2}} + \Phi({}_{k-1}m_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}},e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]})$.

OR2 $:= h^{k-1}_{y_{t_1,\dots,t_{n-1},s_n,\dots,s_{2n-2}}} \cdot x_{s_1,\dots,s_{n-1}} - {}_{k-1}y_{t_1,\dots,t_{n-1}} \cdot g_{s_n,\dots,s_{2n-2},s_1,\dots,s_{n-1}}$ reduces to 0 modulo G using the Leibniz identity of $\mathcal{L}^{\otimes(n-1)}$.

Case 3. Denote by $h_{z_{s_1,\dots,s_{n-1},s_n,\dots,s_{2n-2}}}$ the element $-x_{s_n,\dots,s_{2n-2}} \cdot z_{s_1,\dots,s_{n-1}} + z_{s_1,\dots,s_{n-1}} \cdot x_{s_n,\dots,s_{2n-2}} + \Phi(r_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}},e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}}]})$.

OR3 $:= h_{z_{t_1,\dots,t_{n-1},s_n,\dots,s_{2n-2}}} \cdot x_{s_1,\dots,s_{n-1}} - z_{t_1,\dots,t_{n-1}} \cdot g_{s_n,\dots,s_{2n-2},s_1,\dots,s_{n-1}}$ reduces to 0 modulo G using the Leibniz identity of $\mathcal{L}^{\otimes(n-1)}$.

Case 4. Denote by $t^{k-1}_{y_{s_1,\dots,s_{2n-2}}}$ the element $x_{s_2,\dots,s_n} \cdot {}_{k-1}y_{s_1,s_{n+1},\dots,s_{2n-2}} + \sum_{i=2}^{n-1} {}_{i-1}y_{s_1,\dots,\widehat{s_i},\dots,s_n} \cdot {}_{k-1}y_{s_i,s_{n+1},\dots,s_{2n-2}} + z_{s_1,\dots,s_{n-1}} \cdot {}_{k-1}y_{s_n,s_{n+1},\dots,s_{2n-2}} - \Phi({}_{k-1}m_{[e_{s_1},\dots,e_{s_n}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}})$.

OR4 $:= t^{k-1}_{y_{s_1,\dots,s_{2n-2}}} \cdot x_{t_1,\dots,t_{n-1}} - z_{s_1,\dots,s_{n-1}} \cdot h^{k-1}_{y_{s_n,\dots,s_{2n-2},t_1,\dots,t_{n-1}}} = x_{s_2,\dots,s_n} \cdot {}_{k-1}y_{s_1,s_{n+1},\dots,s_{2n-2}} \cdot x_{t_1,\dots,t_{n-1}} + \sum_{i=2}^{n-1} {}_{i-1}y_{s_1,\dots,\widehat{s_i},\dots,s_n} \cdot {}_{k-1}y_{s_i,s_{n+1},\dots,s_{2n-2}} \cdot x_{t_1,\dots,t_{n-1}} - \Phi({}_{k-1}m_{[e_{s_1},\dots,e_{s_n}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}}) \cdot x_{t_1,\dots,t_{n-1}} + z_{s_1,\dots,s_{n-1}} \cdot x_{t_1,\dots,t_{n-1}} \cdot {}_{k-1}y_{s_n,s_{n+1},\dots,s_{2n-2}} - z_{s_1,\dots,s_{n-1}} \cdot \Phi({}_{k-1}m_{[e_{s_n},\dots,e_{s_{2n-2}}]\otimes e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}})$

$\to_G \sum_{i=2}^{n-1} {}_{i-1}y_{s_1,\dots,\widehat{s_i},\dots,s_n} \cdot {}_{k-1}y_{s_i,s_{n+1},\dots,s_{2n-2}} \cdot x_{t_1,\dots,t_{n-1}} - \Phi({}_{k-1}m_{[e_{s_1},\dots,e_{s_n}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}}) \cdot x_{t_1,\dots,t_{n-1}} - z_{s_1,\dots,s_{n-1}} \cdot \Phi({}_{k-1}m_{[e_{s_n}\otimes\cdots\otimes e_{s_{2n-2}},e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}]}) - x_{s_2,\dots,s_n} \cdot \Phi({}_{k-1}m_{[e_{s_1}\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}},e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}]}) - \Phi(r_{[e_{s_1}\otimes\cdots\otimes e_{s_{n-1}},e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}]}) \cdot {}_{k-1}y_{s_n,s_{n+1},\dots,s_{2n-2}} - \sum_{i=2}^{n-1} x_{t_1,\dots,t_{n-1}} \cdot {}_{i-1}y_{s_1,\dots,\widehat{s_i},\dots,s_n} \cdot {}_{k-1}y_{s_i,s_{n+1},\dots,s_{2n-2}} + x_{t_1,\dots,t_{n-1}} \cdot \Phi({}_{k-1}m_{[e_{s_1},\dots,e_{s_n}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}}) - \Phi(l_{[e_{s_2}\otimes\cdots\otimes e_{s_n},e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}]}) \cdot {}_{k-1}y_{s_1,s_{n+1},\dots,s_{2n-2}} \to_G$

$\Phi({}_{k-1}m_{[[e_{s_1},\ldots,e_{s_n}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}},e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}]})+$
$\sum_{i=2}^{n-1}(-\,{}_{i-1}y_{s_1,\ldots,\widehat{s_i},\ldots,s_n}\cdot\Phi({}_{k-1}m_{[e_{s_i},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}})-$
${}_{i-1}y_{s_1,\ldots,\widehat{s_i},\ldots,s_n}\cdot\Phi({}_{k-1}m_{e_{s_i}\otimes[e_{s_{n+1}},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_{n+2}}\otimes\cdots\otimes e_{s_{2n-2}}})-$
${}_{i-1}y_{s_1,\ldots,\widehat{s_i},\ldots,s_n}\cdot\Phi({}_{k-1}m_{e_{s_i}\otimes e_{s_{n+1}}\otimes[e_{s_{n+2}},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_{n+3}}\otimes\cdots\otimes e_{s_{2n-2}}})-\cdots-$
${}_{i-1}y_{s_1,\ldots,\widehat{s_i},\ldots,s_n}\cdot\Phi({}_{k-1}m_{e_{s_i}\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-3}}\otimes[e_{s_{2n-2}},e_{t_1},\ldots,e_{t_{n-1}}]}))-$
$\sum_{i=2}^{n-1}\Phi({}_{i-1}m_{[e_{s_1}\otimes\cdots\otimes\widehat{e_{s_i}}\otimes\cdots\otimes e_{s_n},e_{t_1}\otimes\cdots\otimes e_{t_{n-1}}]})\cdot{}_{k-1}y_{s_i,s_{n+i},\ldots,s_{2n-2}}+$
$\Phi(l_{e_{s_2}\otimes\cdots\otimes e_{s_{n-1}}\otimes[e_{s_n},e_{t_1},\ldots,e_{t_{n-1}}]})\cdot{}_{k-1}y_{s_1,s_{n+1},\ldots,s_{2n-2}}+$
$\sum_{i=2}^{n-1}\Phi({}_{i-1}m_{e_{s_1}\otimes\cdots\otimes\widehat{e_{s_i}}\otimes\cdots\otimes e_{s_{n-1}}\otimes[e_{s_n},e_{t_1},\ldots,e_{t_{n-1}}]})\cdot{}_{k-1}y_{s_i,s_{n+1},\ldots,s_{2n-2}}-$
$\Phi({}_{k-1}m_{[e_{s_1},\ldots,e_{s_{n-1}},[e_{s_n},e_{t_1},\ldots,e_{t_{n-1}}]]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}})+x_{s_2,\ldots,s_n}\cdot$
$\Phi({}_{k-1}m_{e_{s_1}\otimes[e_{s_{n+1}},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_{n+2}}\otimes\cdots\otimes e_{s_{2n-2}}})+$
$\sum_{i=2}^{n-1}\,{}_{i-1}y_{s_1,\ldots,\widehat{s_i},\ldots,s_n}\cdot\Phi({}_{k-1}m_{e_{s_i}\otimes[e_{s_{n+1}},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_{n+2}}\otimes\cdots\otimes e_{s_{2n-2}}})-$
$\Phi({}_{k-1}m_{[e_{s_1},\ldots,e_{s_n}]\otimes[e_{s_{n+1}},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_{n+2}}\otimes\cdots\otimes e_{s_{2n-2}}})+\cdots+x_{s_2,\ldots,s_n}\cdot$
$\Phi({}_{k-1}m_{e_{s_1}\otimes\cdots\otimes e_{s_{2n-3}}\otimes[e_{s_{2n-2}},e_{t_1},\ldots,e_{t_{n-1}}]})+\sum_{i=2}^{n-1}\,{}_{i-1}y_{s_1,\ldots,\widehat{s_i},\ldots,s_n}\cdot$
$\Phi({}_{k-1}m_{e_{s_i}\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-3}}\otimes[e_{s_{2n-2}},e_{t_1},\ldots,e_{t_{n-1}}]})-$
$\Phi({}_{k-1}m_{[e_{s_1},\ldots,e_{s_n}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-3}}\otimes[e_{s_{2n-2}},e_{t_1},\ldots,e_{t_{n-1}}]})-$
$x_{s_2,\ldots,s_n}\cdot\Phi({}_{k-1}m_{[e_{s_1},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}})-x_{s_2,\ldots,s_n}\cdot$
$\Phi({}_{k-1}m_{e_{s_1}\otimes[e_{s_{n+1}},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_{n+2}}\otimes\cdots\otimes e_{s_{2n-2}}})-\cdots-x_{s_2,\ldots,s_n}\cdot$
$\Phi({}_{k-1}m_{e_{s_1}\otimes\cdots\otimes e_{s_{2n-3}}\otimes[e_{s_{2n-2}},e_{t_1},\ldots,e_{t_{n-1}}]})+x_{s_2,\ldots,s_n}\cdot$
$\Phi({}_{k-1}m_{[e_{s_1},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}})+\sum_{i=2}^{n-1}\Phi({}_{i-1}m_{[e_{s_1},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_2}\otimes\cdots\otimes\widehat{e_{s_i}}\otimes\cdots\otimes e_{s_n}})\cdot$
${}_{k-1}y_{s_i,s_{n+1},\ldots,s_{2n-2}}-\Phi({}_{k-1}m_{[[e_{s_1},e_{t_1},\ldots,e_{t_{n-1}}],e_{s_2},\ldots,e_{s_n}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}})+$
$\Phi(l_{[e_{s_2},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_3}\otimes\cdots\otimes e_{s_n}})\cdot{}_{k-1}y_{s_1,s_{n+1},\ldots,s_{2n-2}}+{}_1y_{s_1,s_3,\ldots,s_n}\cdot$
$\Phi({}_{k-1}m_{[e_{s_2},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}})+$
$\sum_{i=3}^{n-1}\Phi({}_{i-1}m_{e_{s_1}\otimes[e_{s_2},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_2}\otimes\cdots\otimes\widehat{e_{s_i}}\otimes\cdots\otimes e_{s_n}})\cdot{}_{k-1}y_{s_i,s_{n+1},\ldots,s_{2n-2}}-$
$\Phi({}_{k-1}m_{[e_{s_1},[e_{s_2},e_{t_1},\ldots,e_{t_{n-1}}],e_{s_3},\ldots,e_{s_n}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}})+\cdots+\Phi(l_{e_{s_2}\otimes\cdots\otimes e_{s_{n-2}}\otimes[e_{s_{n-1}},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_n}})\cdot$
${}_{k-1}y_{s_1,s_{n+1},\ldots,s_{2n-2}}+\sum_{i=2}^{n-2}\Phi({}_{i-1}m_{e_{s_1}\otimes\cdots\otimes\widehat{e_{s_i}}\otimes\cdots\otimes e_{s_{n-2}}\otimes[e_{s_{n-1}},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_n}})\cdot{}_{k-1}y_{s_i,s_{n+1},\ldots,s_{2n-2}}+$
${}_{n-2}y_{s_1,\ldots,s_{n-2},s_n}\cdot\Phi({}_{k-1}m_{[e_{s_{n-1}},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}})-$
$\Phi({}_{k-1}m_{[e_{s_1},\ldots,e_{s_{n-2}},[e_{s_{n-1}},e_{t_1},\ldots,e_{t_{n-1}}],e_{s_n}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}})-\Phi(l_{[e_{s_2},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_3}\otimes\cdots\otimes e_{s_n}})\cdot$
${}_{k-1}y_{s_1,s_{n+1},\ldots,s_{2n-2}}-\Phi(l_{e_{s_2}\otimes[e_{s_3},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_4}\otimes\cdots\otimes e_{s_n}})\cdot{}_{k-1}y_{s_1,s_{n+1},\ldots,s_{2n-2}}-\cdots-$
$\Phi(l_{e_{s_2}\otimes\cdots\otimes e_{s_{n-2}}\otimes[e_{s_{n-1}},e_{t_1},\ldots,e_{t_{n-1}}]\otimes e_{s_n}})\cdot{}_{k-1}y_{s_1,s_{n+1},\ldots,s_{2n-2}}-\Phi(l_{e_{s_2}\otimes\cdots\otimes e_{s_{n-1}}\otimes[e_{s_n},e_{t_1},\ldots,e_{t_{n-1}}]})\cdot$
${}_{k-1}y_{s_1,s_{n+1},\ldots,s_{2n-2}}=0$ by the fundamental identity of $\mathcal{L}$.

Case 5. Denote by $t_{z_{s_1,\ldots,s_{2n-2}}}$ the element $x_{s_2,\ldots,s_n}\cdot z_{s_1,s_{n+1},\ldots,s_{2n-2}}+\sum_{i=2}^{n-1}\,{}_{i-1}y_{s_1,\ldots,\widehat{s_i},\ldots,s_n}\cdot z_{s_i,s_{n+1},\ldots,s_{2n-2}}+z_{s_1,\ldots,s_{n-1}}\cdot z_{s_n,s_{n+1},\ldots,s_{2n-2}}-\Phi(r_{[e_{s_1},\ldots,e_{s_n}]\otimes e_{s_{n+1}}\otimes\cdots\otimes e_{s_{2n-2}}})$.
$\mathrm{OR5}:=t_{z_{s_1,\ldots,s_{2n-2}}}\cdot x_{j_1,\ldots,j_{n-1}}-z_{s_1,\ldots,s_{n-1}}\cdot h_{z_{s_n,\ldots,s_{2n-2},j_1,\ldots,j_{n-1}}}$.

Case 6. $\mathrm{OR6}:=t_{z_{s_1,\ldots,s_{2n-2}}}\cdot z_{j_1,\ldots,j_{n-1}}-z_{s_1,\ldots,s_{n-1}}\cdot t_{z_{s_n,\ldots,s_{2n-2},j_1,\ldots,j_{n-1}}}$.

Case 7. $\mathrm{OR7}:=t_{z_{s_1,\ldots,s_{2n-2}}}\cdot{}_{k-1}y_{j_1,\ldots,j_{n-1}}-z_{s_1,\ldots,s_{n-1}}\cdot t^{k-1}_{y_{s_n,\ldots,s_{2n-2},j_1,\ldots,j_{n-1}}}$. □

As a consequence of the previous proposition, we obtain

Theorem 15 (*Poincaré–Birkhoff–Witt Theorem*). *Let $\mathcal{L}$ be a Leibniz n-algebra of dimension d. Then a K-basis of the universal enveloping algebra* $\mathrm{U_nL}(\mathcal{L})$ *is formed by the monomials of the type*

$$x_{11\ldots1}^{a_{11\ldots1}}\cdots\widehat{x}_{\alpha_1\ldots\alpha_{n-1}}^{a_{\alpha_1\ldots\alpha_{n-1}}}\cdot h({}_1y_{11\ldots1},\ldots,{}_{n-2}y_{dd\ldots d})\cdot z^e_{s_1,\ldots,s_{n-1}},$$

where $h({}_1y_{11\ldots1},\ldots,{}_{n-2}y_{dd\ldots d})$ is a monic monomial in ${}_1y_{11\ldots1},\ldots,{}_{n-2}y_{dd\ldots d}$ and $e=0,1$.

Example 16. Let $\mathcal{L}$ be a Leibniz 3-algebra of dimension 2 with basis $\{e_1, e_2\}$ and the 3-bracket defined by $[e_i, e_j, e_i] := e_j$, $[e_i, e_i, e_j] := -e_j$, if $i \neq j$, and 0 in other case. A basis of $(\mathcal{L} \otimes \mathcal{L})^{\mathrm{ann}}$ is $\{e_2 \otimes e_1 + e_1 \otimes e_2, e_2 \otimes e_2 - e_1 \otimes e_1\}$. Identifying

$$\mathrm{T}\left((\mathcal{L} \otimes \mathcal{L})^l \oplus (\mathcal{L} \otimes \mathcal{L})^m \oplus (\mathcal{L} \otimes \mathcal{L})^r\right)$$

with $K{<}x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}, z_{11}, z_{12}, z_{21}, z_{22}{>}$, we obtain $g_1 = x_{21} + x_{12}$, $g_2 = x_{22} - x_{11}$, $\alpha_1\alpha_2 = 21$, $X_1 = \{x_{11}, x_{12}\}$ and $X_2 = \{x_{21}, x_{22}\}$.

A minimal Gröbner basis of the ideal $\Phi(I)$, with respect to the degree lexicographical ordering $x_{11} < x_{12} < x_{21} < x_{22} < y_{11} < y_{12} < y_{21} < y_{22} < z_{11} < z_{12} < z_{21} < z_{22}$, by Proposition 14, is $G = \{x_{kt} \cdot x_{ij} - x_{ij} \cdot x_{kt} - \Phi(l_{[e_i \otimes e_j, e_k \otimes e_t]})\}_{(i,j)<(k,t), x_{kt} \in X_1} \cup \{x_{21} + x_{12}, x_{22} - x_{11}\}$ $\cup \{x_{kt} \cdot y_{ij} - y_{ij} \cdot x_{kt} - \Phi(m_{[e_i \otimes e_j, e_k \otimes e_t]}), x_{kt} \cdot z_{ij} - z_{ij} \cdot x_{kt} - \Phi(r_{[e_i \otimes e_j, e_k \otimes e_t]})\}_{x_{kt} \in X_1}$ $\cup \{x_{jk} \cdot y_{it} + y_{ik} \cdot y_{jt} + z_{ij} \cdot y_{kt} - \Phi(m_{[e_i, e_j, e_k] \otimes e_t}), x_{jk} \cdot z_{it} + y_{ik} \cdot z_{jt} + z_{ij} \cdot z_{kt} - \Phi(r_{[e_i, e_j, e_k] \otimes e_t}))\}$, with $i, j, k, t \in \{1, 2\}$.

In this case G has 51 polynomials, where $\{x_{kt} \cdot x_{ij} - x_{ij} \cdot x_{kt} - \Phi(l_{[e_i \otimes e_j, e_k \otimes e_t]})\}_{(i,j)<(k,t), x_{kt} \in X_1}$ $= \{x_{12} \cdot x_{11} - x_{11} \cdot x_{12} + x_{21} + x_{12}\}$ and $\{x_{kt} \cdot y_{ij} - y_{ij} \cdot x_{kt} - \Phi(m_{[e_i \otimes e_j, e_k \otimes e_t]}), x_{kt} \cdot z_{ij} - z_{ij} \cdot x_{kt} - \Phi(r_{[e_i \otimes e_j, e_k \otimes e_t]})\}_{x_{kt} \in X_1} = \{x_{11} \cdot y_{11} - y_{11} \cdot x_{11}, x_{11} \cdot y_{12} - y_{12} \cdot x_{11}, x_{11} \cdot y_{21} - y_{21} \cdot x_{11}, x_{11} \cdot y_{22} - y_{22} \cdot x_{11}, x_{12} \cdot y_{11} - y_{11} \cdot x_{12} + y_{21} + y_{12}, x_{12} \cdot y_{12} - y_{12} \cdot x_{12} + y_{22} - y_{11}, x_{12} \cdot y_{21} - y_{21} \cdot x_{12} - y_{11} + y_{22}, x_{12} \cdot y_{22} - y_{22} \cdot x_{12} - y_{12} - y_{21}, x_{11} \cdot z_{11} - z_{11} \cdot x_{11}, x_{11} \cdot z_{12} - z_{12} \cdot x_{11}, x_{11} \cdot z_{21} - z_{21} \cdot x_{11}, x_{11} \cdot z_{22} - z_{22} \cdot x_{11}, x_{12} \cdot z_{11} - z_{11} \cdot x_{12} + z_{21} + z_{12}, x_{12} \cdot z_{12} - z_{12} \cdot x_{12} + z_{22} - z_{11}, x_{12} \cdot z_{21} - z_{21} \cdot x_{12} - z_{11} + z_{22}, x_{12} \cdot z_{22} - z_{22} \cdot x_{12} - z_{12} - z_{21}\}$.

G is minimal but not reduced since there are in this list polynomials which are not in the normal form, e.g., $x_{12} \cdot x_{11} - x_{11} \cdot x_{12} + x_{21} + x_{12}$.

Thus, a K-basis of $\mathrm{U_3L}(\mathcal{L})$ is formed by the monomials of the type

$$x_{11}^a \cdot x_{12}^b \cdot h(y_{11}, y_{12}, y_{21}, y_{22}),$$
$$x_{11}^a \cdot x_{12}^b \cdot h(y_{11}, y_{12}, y_{21}, y_{22}) \cdot z_{11},$$
$$x_{11}^a \cdot x_{12}^b \cdot h(y_{11}, y_{12}, y_{21}, y_{22}) \cdot z_{12},$$
$$x_{11}^a \cdot x_{12}^b \cdot h(y_{11}, y_{12}, y_{21}, y_{22}) \cdot z_{21},$$
$$x_{11}^a \cdot x_{12}^b \cdot h(y_{11}, y_{12}, y_{21}, y_{22}) \cdot z_{22}.$$

5. Implementation of calculations

In this section we describe a program in NCAlgebra (Helton et al., 1996) (a package running under Mathematica) for implementing the algorithms discussed in this paper. The program will calculate the reduced Gröbner basis of the ideal $\Phi(I)$ that determines $\mathrm{U_nL}(\mathcal{L})$ for small values of n and low dimensions of $\mathcal{L}$. The Mathematica code, together with some examples, is also available in http://web.usc.es/~mladra/research.html.

```
##############################################################################

 (* This program computes the reduced Gröbner Basis of the ideal Φ(I). To
    run properly this code it is necessary to load the NCGB package *)

##############################################################################
```

```
(* Let be ℒ a Leibniz n-algebra of dimension d *)
n= (* Write here the value of n *)
d= (* Write here the value of d *)
(* Insert the Bracket represented by Bracket[{s1,...,sn}] := {a1,...,ad} where
[e_s1,...,e_sn] = a1 e1+···+ad ed. In Example 16, e.g., Bracket[{1, 1, 2}]:={0,-1} *)
(* Relations are generated *)
(* Type (1) relations *)

RelationsOne[IndexSet_List, k_] :=
    x[Take[IndexSet, {2, n}]] **
      y[Join[{k - 1},{IndexSet[[1]]},Take[IndexSet, {n + 1, 2*n - 2}]]] +
      Sum[y[Join[{i - 1}, Take[IndexSet, {1, i - 1}],
        Take[IndexSet, {i + 1, n}]]] ** y[Join[{k - 1}, {IndexSet[[i]]},
         Take[IndexSet, {n + 1, 2*n - 2}]]], {i, 2, n - 1}] +
      z[Take[IndexSet, {1, n - 1}]] **
      y[Join[{k - 1}, Take[IndexSet, {n, 2*n - 2}]]] -
      Bracket[Take[IndexSet, {1, n}]].Table[y[Join[{k - 1}, {i},
         Take[IndexSet, {n + 1, 2*n - 2}]]], {i, 1,d}]}

(* Type (2) relations *)

RelationsTwo[IndexSet_List] :=
    x[Take[IndexSet, {2, n}]] **
      z[Join[{IndexSet[[1]]}, Take[IndexSet, {n + 1, 2*n - 2}]]] +
      Sum[y[Join[{i - 1}, Take[IndexSet, {1, i - 1}],
      Take[IndexSet, {i + 1, n}]]] ** z[Join[{IndexSet[[i]]},
      Take[IndexSet, {n + 1, 2*n - 2}]]], {i, 2,n- 1}] +
      z[Take[IndexSet, {1, n - 1}]] ** z[Take[IndexSet, {n, 2*n - 2}]] -
      Bracket[Take[IndexSet, {1, n}]].Table[z[Join[{i},
      Take[IndexSet, {n + 1, 2*n - 2}]]], {i, 1, d}]

(* Linear Components of relations (3), (4) and (5) *)

LinearComponents[IndexSet_List, var_List] :=
    Module[{target},
      target = {};
      target = Sum[ Bracket[Join[{IndexSet[[i]]},
         Take[IndexSet, {n, 2*n - 2}]]].
         Table[var[[1]][Join[var[[2]], Take[IndexSet, {1, i - 1}], {j},
         Take[IndexSet, {i + 1, n - 1}]]], {j, 1, d}], {i, 1, n - 1}];
      Return[target]; ]

(* Type (3) relations *)

RelationsThree[IndexSet_List] :=
  x[Take[IndexSet, {n, 2*n - 2}]] ** x[Take[IndexSet, {1, n - 1}]] -
    x[Take[IndexSet, {1, n - 1}]] ** x[Take[IndexSet, {n, 2*n - 2}]] -
    LinearComponents[IndexSet, {x, {}}]
```

```
(* Type (4) relations *)

RelationsFour[IndexSet_List, k_] :=
  x[Take[IndexSet, {n, 2*n - 2}]] **y[Join[{k - 1},
   Take[IndexSet, {1, n - 1}]]] - y[Join[{k - 1},
     Take[IndexSet, {1, n - 1}]]] ** x[Take[IndexSet, {n, 2*n - 2}]] -
    LinearComponents[IndexSet, {y, {k - 1}}]

(* Type (5) relations *)

RelationsFive[IndexSet_List] :=
  x[Take[IndexSet, {n, 2*n - 2}]] ** z[Take[IndexSet, {1, n - 1}]] -
    z[Take[IndexSet, {1, n - 1}]] ** x[Take[IndexSet, {n, 2*n - 2}]] -
    LinearComponents[IndexSet, {z, {}}]

(* Generators of the ideal Φ(I) *)

G = {}
A = Tuples[Table[i, {i, 1, d}], 2*n - 2]
lengA = Length[A]

(* All type (1) relations are generated *)

Do[ Do[G = Join[G,{RelationsOne[A[[i]], k]}], {i, 1, lengA}],{k,2,n - 1}]

(* All type (2) relations are generated *)

Do[G = Join[G, {RelationsTwo[A[[i]]]}] , {i, 1, lengA}]

(* All type (3) relations are generated *)

Do[G = Join[G, {RelationsThree[A[[i]]]}] , {i, 1, lengA}]

(* All type (4) relations are generated *)

Do[ Do[G = Join[G,{RelationsFour[A[[i]],k]}],{i,1,lengA}],{k,2,n-1}]

(* All type (5) relations are generated *)

Do[ G = Join[G, {RelationsFive[A[[i]]]}] , {i, 1, lengA}]

(* Noncommutative variables are defined *)

Variabs = {}
B = Tuples[Table[i, {i, 1, d}], n - 1]
lengtB=Length[B]
Do[Variabs = Join[Variabs, {x[B[[i]]]}]; , {i, 1, lengtB}]
Do[ Do[Variabs = Join[Variabs, {y[Join[{k - 1}, B[[i]]]]}];
    , {i, 1, lengtB}], {k, 2, n - 1}]
Do[Variabs = Join[Variabs, {z[B[[i]]]}];, {i, 1, lengtB}]
```

```
SetNonCommutative @@ Variabs
SetMonomialOrder[Variabs]
NCMakeGB[G,10]

      (* This output gives the normal forms of the polynomials
         which appear in Proposition 14 *)
```

Acknowledgements

This research was partly supported by MEC (Spain), Grant MTM2006-15338-C02 (European FEDER support included) and project Ingenio Mathematica (i-MATH) No. CSD2006-00032 (Consolider Ingenio 2010) and by Xunta de Galicia, PGIDITI06PXIB371128PR.

References

Aymon, M., 1997. Algèbre enveloppante et digèbre enveloppante d'une algèbre de Leibniz. Travail de diplôme. Université de Genève.

Bergman, G.M., 1978. The diamond lemma for ring theory. Adv. Math. 29, 178–218.

Cartan, E., 1952. Oeuvres complètes. Partie I. Groupes de Lie. Gauthier-Villars, Paris.

Casas, J.M., 2006. Trialgebras and Leibniz 3-algebras. Bol. Soc. Mat. Mexicana 12 (2), 165–178.

Casas, J.M., Loday, J.-L., Pirashvili, T., 2002. Leibniz n-algebras. Forum Math. 14, 189–207.

Daletskii, Y.I., Takhtajan, L.A., 1997. Leibniz and Lie algebra structures for Nambu algebra. Lett. Math. Phys. 39, 127–141.

Filippov, V.T., 1985. n-Lie algebras. Sib. Mat. Zh. 26 (6), 126–140.

Green, E.L., 1994. An introduction to noncommutative Gröbner bases. In: Computational Algebra. In: Lect. Notes in Pure and Appl. Math., vol. 151. Marcel Dekker, Inc., New York, pp. 167–190.

Hagiwara, Y., Mizutani, T., 2002. Leibniz algebras associated with foliations. Kodai Math. J. 25, 151–165.

Harris, B., 1961. Cohomology of Lie triple systems and Lie algebras with involution. Trans. Amer. Math. Soc. 98, 148–162.

Helton, J.W., Miller, R.L., Stankus, M., 1996. NCAlgebra: A mathematica package for doing noncommuting algebra. Available from: http://math.ucsd.edu/˜ncalg.

Hodge, T.L., Parshall, B.J., 2002. On the representation theory of Lie triple systems. Trans. Amer. Math. Soc. 354, 4359–4391.

Ibáñez, R., de León, M., Marrero, J.C., Padrón, E., 1999. Leibniz algebroid associated with a Nambu–Poisson structure. J. Phys. A 32, 8129–8144.

Insua, M.A., Ladra, M., 2007. Gröbner bases in enveloping algebras of Leibniz algebras. J. Symbolic Comput. (submitted for publication).

Jacobson, N., 1949. Lie and Jordan triple systems. Amer. J. Math. 71, 149–170.

Kinyon, M.K., Weinstein, A., 2001. Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces. Amer. J. Math. 123, 525–550.

Lister, W., 1952. A structure theory of Lie triple systems. Trans. Amer. Math. Soc. 72, 217–242.

Loday, J.-L., 1993. Une version non commutative des algèbres de Lie: Les algèbres de Leibniz. Enseign. Math. 3, 269–293.

Loday, J.-L., Pirashvili, T., 1993. Universal enveloping algebras of Leibniz algebras and (co)homology. Math. Ann. 296, 139–158.

Mora, T., 1994. An introduction to commutative and noncommutative Gröbner bases. Theoret. Comput. Sci. 134, 131–173.

Nambu, Y., 1973. Generalized Hamiltonian dynamics. Phys. Rev. D 7 (3), 2405–2412.

Takhtajan, L., 1994. On foundation of the generalized Nambu mechanics. Comm. Math. Phys. 160, 295–315.